1 SUMMARY

To compute values of the **natural logarithm of the Gamma function,** $w = \ln|\Gamma(z)|$ **for complex argument** z; where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad Rz > 0$$

The principal value of the multivalued function $\ln \Gamma(z)$ is computed, taking $\arg z = \pi$ for z real and negative. The calculation is performed to high precision over a wide range of argument values using FC14 and an asymptotic expansion from Abramowitz and Stegun, Handbook of Mathematical Functions.

ATTRIBUTES — **Version:** 1.0.0. **Types:** FC15A; FC15AD. **Calls:** FC14. **Language:** FC15AD uses COMPLEX*16. **Original date:** November 1982. **Origin:** A.R.Curtis, Harwell.

2 HOW TO USE THE PACKAGE

2.1 Computing Gamma Function Values

The logarithm of $\Gamma(z)$ is the natural function to use, as $\Gamma(z)$ itself is harder to compute and much more likely to underflow or overflow. If $\Gamma(z)$ is required (and will not overflow), it can be obtained by using the Fortran complex exponential function subroutine CDEXP after return from FC15, then truncating the result to single precision if desired. Note that the double precision version FC15AD is the basic one; if the single precision version is called, it copies the x and y values into double precision variables and calls FC15AD, and truncates the double precision results to single precision, thus giving full single precision accuracy.

2.2 The Argument List and Calling Sequence

The single precision version

```
CALL FC15A(X,Y,U,V)
```

The double precision version

```
CALL FC15AD(X,Y,U,V)
```

X is a REAL (DOUBLE PRECISION in the D version) variable which must be set by the user to x the real part of the value of z=x+iy for which the function is to be calculated. This argument is not altered by the subroutine. **Restrictions:** if y is zero, x must not be a negative integer, zero, or close enough to any of these values so as to cause underflow.

Y is a REAL (DOUBLE PRECISION in the D version) variable which must be set by the user to y the imaginary part of the value of z=x+iy for which the function is to be calculated. This argument is not altered by the subroutine.

U is a REAL (DOUBLE PRECISION in the D version) variable which will be set by the subroutine to the computed value u of the real part of the function value

$$w = u + iv = \ln \Gamma(z).$$

V is a REAL (DOUBLE PRECISION in the D version) variable which will be set by the subroutine to the computed value v of the imaginary part of the function value w=u+iv (see argument U).

3 GENERAL INFORMATION

Use of common: none.

Workspace: none.

Other routines called directly: calls `FC14`.

Input/output: none.

Restrictions:

$z \neq$ negative integer,
$z \neq 0$,

or close enough to any of these values to cause underflow.

Portability: `FC15` uses `COMPLEX*16` facility.

Accuracies:

6 figures using 4-byte arithmetic

$< \max\{1, \ln|\Gamma(x)|\} \times 10^{-15}$ using 8-byte arithmetic.

4 METHOD

If y=0, the library subroutine `FC14` is called to set u to the value of $\ln|\Gamma(x)|$, and then v is set to zero if $x > 0$, or to $-n\pi$ if $-n < x \le 1-n$. Note that `FC14` will fail if $x=-n$, n=0,1,2,....

Otherwise, if $x \ge 0$ a complex variable $z_1 = z+m$, where $m > 0$ is a real integer, is constructed such that $\mathrm{Re}(z_1) \ge 1$ and $|z_1| \ge 6$. The asymptotic series expansion (Abramowitz and Stegun, Handbook of Mathematical Functions, 6.1.40)

$$\ln\Gamma(z_1) \approx (z_1 - \tfrac{1}{2}) \ln z_1 - z_1 + \tfrac{1}{2}\ln 2\pi + \sum_{r=1}^{10} \frac{a_r}{z_1^{2r-1}} \tag{1}$$

is used in the modified form

$$\ln\Gamma(z) \approx (z_1 - \tfrac{1}{2}) \ln z_1 - z_1 + \tfrac{1}{2}\ln 2\pi + \sum_{r=1}^{10} \frac{a_r}{z_1^{2r-1}} - \ln\left\{\prod_{s=1}^{m}\left(\frac{z_1 - s}{z_1}\right)\right\} \tag{2}$$

(the final term being omitted for m=0.)

If $x < 0$, a complex variable $z_2 = 1-z$ is formed, and $\ln\Gamma(z_2)$ found as above; then the formula

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z \tag{3}$$

is used in the form

$$\begin{aligned} \ln\Gamma(z) &= -\ln\Gamma(z_2) - \ln\left(\frac{\sin \pi z}{\pi}\right) \\ &= \ln 2\pi - \pi|y| + i(\tfrac{1}{2} - x)\mathrm{sign}(y) - \\ &\quad - \ln\Gamma(z_2) - ln\{1 - \exp 2\pi \mathrm{sign}(y)(it - y)\} \end{aligned} \tag{4}$$

where $t=x+n$ is such that $0 \le t < 1$. This form avoids overflows in the complex sine subroutine for $|y|$ large; underflows in the exponential are avoided by dropping the last term if $|y| \ge 6$, since $\exp(-12\pi)$ is negligible to the required accuracy. Dropping this last term also saves execution time in cases to which it is applies.

This method ensures that the correct branch of the multivalued function $\ln\Gamma(z)$ is used, with discontinuities of integral multiples of $2\pi i$ only across the negative real axis. These arise because $\Gamma(z)$ has poles, so that $\ln\Gamma(z)$ has branch points, at z=0,–1,–2,....

The asymptotic series has a maximum error of about 10^{-16}. The result should therefore be within the rounding errors of the double precision complex logarithm and (for (4)) exponential functions, and the accumulated arithmetical rounding errors, including some cancellation in the square brackets in (2).

FC15 was tested on an `IBM/3081` computer by computing all the values (1111 12-decimal complex numbers) tabulated in Table 6.7 of Abramwitz and Stegun, which for this purpose we treat as correctly rounded from exact values. Agreement was exact except for a total of seven individual real or imaginary parts, which each differed by a unit in the 12th decimal place. This is what would be expected by chance if root mean square errors were about 3.6×10^{-15} in each part, i.e. about 5.1×10^{-15} in absolute value. One would expect the errors to be of order $\varepsilon\times\max(1,|w|)$ for some ε; since in all cases of discrepancy $|w|\geq 5.5$, we conclude that $\varepsilon< 10^{-15}$, which is about what would be expected on the `IBM/3081` computer. This covers arguments in the rectangle $1\leq x\leq 2$, $0\leq y\leq 10$; there is no reason to suppose that accuracy is further degraded outside this triangle.