1 SUMMARY

Given \( n \) values of a function \( f(x) \) calculates the \( m \)th degree minimax polynomial approximation \( P(x) \), where \( n \geq m \), and \( 2 \leq m \leq 25 \), such that

\[
\max_{1 \leq k \leq n} | f(x_k) - P(x_k) |
\]

is minimized.


2 HOW TO USE THE PACKAGE

2.1 The argument list and calling sequence

The single precision version:

\[
\text{CALL PE11A}(F,X,N,M,NE,I,A,NC)
\]

The double precision version:

\[
\text{CALL PE11AD}(F,X,N,M,NE,I,A,NC)
\]

\( F \) is a REAL (DOUBLE PRECISION in the D version) array of length at least \( n \), which must be set by the user to the function values so that \( F(j) = f(x_j) \).

\( X \) is a REAL (DOUBLE PRECISION in the D version) array of length at least \( n \), which must be set by the user to the points \( x \) so that \( X(j) = x_j \).

\( N \) is an INTEGER which must be set by the user to the number of points.

\( M \) is an INTEGER variable which must be set by the user to the maximum degree that the approximating \( \ast \) polynomial \( p(x) \) will be allowed to take.

\( NE \) is an INTEGER variable which is an error return parameter which is set to zero if \( M < N \), and is set to 1 otherwise.

\( I \) is an INTEGER array of length at least \( M+4 \) which will be set by the subroutine to the positions of the \( M+2 \) reference points so that the \( k \)th reference point is \( X(I(K+1)) \).

\( A \) is a REAL (DOUBLE PRECISION in the D version) array of length at least \( M+1 \) set by the subroutine to the coefficients of the polynomial \( p(x) \) so that \( A(J+1) = a_{j+1} \) where

\[
p(x) = \sum_{j=0}^{m} a_{j+1} x^j
\]

\( NC \) is an INTEGER variable which controls output from the subroutine. The basic output from PE11A/AD is the array \( A \) defining the polynomial. If no other output is required then set \( NC=3 \).

If \( NC=1 \) or 2, then a table of the values \( x_i, f(x_i), p^\ast(x_i), f(x_i) - p^\ast(x_i) \) is printed.

If \( NC=2 \) or 4, then the vector \( u \) is identified by printing the subscripts \( i_1, i_2, \ldots, i_{M+2} \) where \( u = (x_{i_1}, x_{i_2}, \ldots, x_{i_{M+2}}) \).

With this value of \( NC \), \( \delta \) is also printed.

If \( NC \leq 0 \) then all printing is suppressed.
2.2 The common block.

A labelled common block PE11B/BD is used.

The single precision version

COMMON/PE11B/LP,LPD

The double precision version

COMMON/PE11BD/LP,LPD

LP is an INTEGER variable set to the stream number for printing of results. (default value is 6).

LPD is an INTEGER variable set to the stream number for diagnostic printing. (default value is 6).

3 GENERAL INFORMATION

Use of Common: a COMMON block PE11B/BD is used, see § 2.2.

Workspace: none.

Other subroutines: none

Input/Output: see § 2.1.

4 METHOD

The method used is described in P.C.Curtiss and W.L.Frank, Journal of A.C.M. 1959, pp.395-404. If \( \mathbf{u} = (u_0,u_1,\ldots,u_n) \) where \( u_i \) is some point of the set \( x_1,x_2,\ldots,x_n \) consider the equations

(i) \( a_0u_0x_0^k + a_1u_1x_1^k + \ldots + a_nu_nx_n^k + (-1)^k \delta = f(u_i); k=1,2,\ldots \)

(ii) \( |\delta| = \max_{1\leq i \leq n}[|f(x_i)+(a_0u_0x_i^k + a_1u_1x_i^k + \ldots + a_nu_nx_n^k)] \)

Then (i) can be solved for \( a_0,a_1,\ldots,a_n,\delta \). There is a \( \mathbf{u} \) such that the solution of (i) satisfies (ii). If \( q(x) \) is the polynomial with coefficients \( a_0,a_1,\ldots,a_n \) then \( p'(x) = q(x) \). Note that \( \mathbf{u} \) is not necessarily unique.

5 EXAMPLE OF USE

Suppose we have a table of points with corresponding function values and wish to find an approximating polynomial to the function. Then code of the following form could be used.

```
DOUBLE PRECISION F(20),X(20),A(10)
INTEGER II(10)
DATA F( 1)/ 0.98322D+04/,F( 2)/ 0.59127D+04/,F( 3)/ 0.32993D+04/, ...
DATA X( 1)/-0.85000D+01/,X( 2)/-0.75000D+01/,X( 3)/-0.65000D+01/, ...
C NUMBER OF FUNCTION VALUES.
N=20
```
C  MAXIMUM DEGREE OF APPROXIMATING POLYNOMIAL.
M=4
C  PRINT TABLE OF VALUES.
NC=1
CALL PE11AD(F,X,N,M,NE,II,A,NC)
WRITE(6,20)(A(I),I=1,5)

20 FORMAT(© THE COEFFICIENTS OF THE POLYNOMIAL \',/,,
+ ' A(1)+A(2)*X+A(3)*X**2+A(4)*X**3+A(5)*X**4 ARE \',/,,
+ ' A(1)=',D14.6,' A(2)=',D14.6,' A(3)=',D14.6,' A(4)=',D14.6,' A(5)=',D14.6)
STOP
END

The following output would be produced.

<table>
<thead>
<tr>
<th>I</th>
<th>X</th>
<th>F(X )</th>
<th>P(X )</th>
<th>F(X )-P(X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.850000E+01</td>
<td>0.98322000E+04</td>
<td>0.98324707E+04</td>
<td>-0.27073554E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.750000E+01</td>
<td>0.59127000E+04</td>
<td>0.59117070E+04</td>
<td>0.99297924E+00</td>
</tr>
<tr>
<td>3</td>
<td>-0.650000E+01</td>
<td>0.32993000E+04</td>
<td>0.32995797E+04</td>
<td>-0.65797720E+00</td>
</tr>
<tr>
<td>4</td>
<td>-0.550000E+01</td>
<td>0.16679000E+04</td>
<td>0.16672174E+04</td>
<td>0.68255098E+00</td>
</tr>
<tr>
<td>5</td>
<td>-0.450000E+01</td>
<td>0.73080000E+03</td>
<td>0.73148041E+03</td>
<td>-0.68041310E+00</td>
</tr>
<tr>
<td>6</td>
<td>-0.350000E+01</td>
<td>0.25775000E+03</td>
<td>0.25842975E+03</td>
<td>-0.99297924E+00</td>
</tr>
<tr>
<td>7</td>
<td>-0.250000E+01</td>
<td>0.62912000E+02</td>
<td>0.63002390E+02</td>
<td>-0.90389996E-01</td>
</tr>
<tr>
<td>8</td>
<td>-0.150000E+01</td>
<td>0.72500000E+02</td>
<td>0.62570207E+01</td>
<td>0.99297924E+00</td>
</tr>
<tr>
<td>9</td>
<td>-0.500000E+00</td>
<td>-0.98339000E-00</td>
<td>-0.14936202E+01</td>
<td>0.51023026E+00</td>
</tr>
<tr>
<td>10</td>
<td>0.500000E+00</td>
<td>-0.27854000E+01</td>
<td>-0.22488920E+01</td>
<td>0.53650794E+00</td>
</tr>
<tr>
<td>11</td>
<td>0.150000E+01</td>
<td>0.90000000E+01</td>
<td>0.99297924E+00</td>
<td>-0.99297924E+00</td>
</tr>
<tr>
<td>12</td>
<td>0.250000E+01</td>
<td>0.88816000E+02</td>
<td>0.89234900E+02</td>
<td>-0.41890028E+00</td>
</tr>
<tr>
<td>13</td>
<td>0.350000E+01</td>
<td>0.33670000E+03</td>
<td>0.33748091E+03</td>
<td>0.52908944E+00</td>
</tr>
<tr>
<td>14</td>
<td>0.450000E+01</td>
<td>0.90512000E+03</td>
<td>0.90473618E+03</td>
<td>0.38381766E+00</td>
</tr>
<tr>
<td>15</td>
<td>0.550000E+01</td>
<td>0.19900000E+04</td>
<td>0.19890070E+04</td>
<td>0.99297924E+00</td>
</tr>
<tr>
<td>16</td>
<td>0.650000E+01</td>
<td>0.38360000E+04</td>
<td>0.38363008E+04</td>
<td>-0.30086374E+00</td>
</tr>
<tr>
<td>17</td>
<td>0.750000E+01</td>
<td>0.67405000E+04</td>
<td>0.67406262E+04</td>
<td>-0.12628205E+00</td>
</tr>
<tr>
<td>18</td>
<td>0.850000E+01</td>
<td>0.11043000E+05</td>
<td>0.11043992E+05</td>
<td>-0.99297924E+00</td>
</tr>
<tr>
<td>19</td>
<td>0.950000E+01</td>
<td>0.17136000E+05</td>
<td>0.17136411E+05</td>
<td>-0.41179170E+00</td>
</tr>
<tr>
<td>20</td>
<td>1.050000E+02</td>
<td>0.25456000E+05</td>
<td>0.25455894E+05</td>
<td>0.10531335E+00</td>
</tr>
</tbody>
</table>

THE COEFFICIENTS OF THE POLYNOMIAL
A(1)+A(2)*X+A(3)*X**2+A(4)*X**3+A(5)*X**4 ARE
A(1)= -0.199576D+01 A(2)= -0.100535D+01 A(3)= -0.198992D-02
A(4)= 0.100030D+01 A(5)= 0.200005D+01